Coleman integration for general curves

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Joint work with Jennifer Balakrishnan (Boston University)
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- $P, Q \in X(Q_p)$,
- $\omega \in \Omega^1(X)$.
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\[ \int_P^Q \omega. \]

Actually, we can
- replace $\mathcal{Q}_p$ and $\mathcal{Z}_p$ by $\mathcal{C}_p$ and its valuation ring,
- take $\omega \in \Omega^1(U)$ for some (wide) open $U \subset X^{(an)}$,
- extend to integrate over $D \in J(\mathcal{Q}_p)$ where $J$ is the Jacobian of $X$ (above: $D = Q - P$).
The Coleman integral satisfies (and is characterised by) the following properties;

**Theorem**

1. **Linearity:** \( \int_P^Q \alpha \omega_1 + \beta \omega_2 = \alpha \int_P^Q \omega_1 + \beta \int_P^Q \omega_2. \)
2. **Additivity:** \( \int_P^R \omega = \int_P^Q \omega + \int_Q^R \omega. \)
3. **Change of variables:** \( \int_P^Q f^* \omega = \int_{f(P)}^{f(Q)} \omega. \)
4. **Fundamental theorem of calculus:** \( \int_P^Q df = f(Q) - f(P). \)

Locally (on disks of radius less than 1) you can define the integral as in complex analysis. However, because the \( p \)-adic topology is totally disconnected, there is no notion of analytic continuation to fix the integration constants.

Coleman’s key idea is to replace analytic continuation by equivariance with respect to the Frobenius map \( F_p \) to move between different disks.
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**Theorem**

Let $\mathcal{X}$ be a curve of genus $g \geq 2$ over $\mathbb{Q}$, $J$ the Jacobian of $\mathcal{X}$, $p$ a prime of good reduction and $X = \mathcal{X} \otimes \mathbb{Q}_p$. Moreover, let $r$ be the Mordell-Weil rank of $\mathcal{X}$ and suppose that $r < g$. Then there exists $\omega \in \Omega^1(X)$ such that $\int_{P}^{Q} \omega = 0$ for all $P, Q$ in $\mathcal{X}(\mathbb{Q})$. 

So by computing Coleman integrals, we might sometimes be able to find rational points, or prove that we have found all of them.

**Remark**

The nonabelian Chabauty program by Kim tries to get rid of the assumption $r < g$. Note that this still involves (iterated) Coleman integrals!
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*The nonabelian Chabauty program by Kim tries to get rid of the assumption $r < g$. Note that this still involves (iterated) Coleman integrals!*
Let $X$ be a hyperelliptic curve of genus $g$ given by

$$y^2 = f(x)$$

with $f(x) \in \mathbb{Z}_p[x]$ monic of degree $2g + 1$ separable mod $p$. 
**Hyperelliptic curves**

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with $f(x) \in \mathbb{Z}_p[x]$ monic of degree $2g + 1$ separable mod $p$.

Balakrishnan, Bradshaw and Kedlaya gave an algorithm (implemented in SAGE) to compute Coleman integrals in this case.

The method is based on Kedlaya’s algorithm for computing zeta functions of hyperelliptic curves over finite fields.

Has been successfully used for doing new cases of effective Chabauty by Balakrishnan and various co-authors.
General curves

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However, recently I have developed and implemented a practical extension of Kedlaya’s algorithm to (almost) all curves.

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It is therefore natural to ask if this algorithm can also be used to compute Coleman integrals on (more) general curves.
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The answer is yes!
$p$-adic cohomology

To compute Coleman integrals, we will be using $p$-adic (also called rigid) cohomology.

For a curve $X/\mathbb{Z}_p$ one can define finite dimensional $\mathbb{Q}_p$ vector spaces $H^i_{\text{rig}}(X)$ (for $i = 0, 1, 2$) that are functorial in (so only depend on) the special fibre $\overline{X} = X \otimes \mathbb{F}_p$. Note that in particular the $p$-th power Frobenius map on $\overline{X}$ acts on the $H^i_{\text{rig}}(X)$.

These $p$-adic cohomology spaces have similar properties as:

- $l$-adic (étale) cohomology (for $l \neq p$)
- crystalline cohomology,

but are much easier to define and compute.

We will now give a rather computational introduction to this theory, keeping everything as explicit as possible, since that is essential for our algorithm.
Setup

Let $\mathcal{X}$ be a nonsingular projective curve of genus $g$ over $\mathbb{Q}$ given by a (possibly singular) plane model $Q(x, y) = 0$ with $Q(x, y) \in \mathbb{Z}[x, y]$ irreducible and monic in the variable $y$.

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$\Delta(x) \in \mathbb{Z}[x]$ the discriminant of $Q(x, y)$ w.r.t. $y$.

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Note that if $r(x_0) = 0$ then one of the following two holds:

- the plane model $Q(x, y)$ has a singularity lying over $x_0$,
- the map $x : \mathcal{X} \to \mathbb{P}^1$ has a ramification point lying over $x_0$. 
Integral basis

Let $\mathbb{Q}(\mathcal{X})$ denote the function field of the curve $\mathcal{X}$.

Definition

We let $W^0 \in \text{Gl}_{d_y}(\mathbb{Q}[x, 1/r])$ denote a matrix such that, if

$$b^0_j = \sum_{i=0}^{d_y-1} W^0_{i+1,j+1} y^i$$

then $[b^0_0, \ldots, b^0_{d_y-1}]$ is an integral basis for $\mathbb{Q}(\mathcal{X})$ over $\mathbb{Q}[x]$.

Similarly, we let $W^\infty \in \text{Gl}_{d_y}(\mathbb{Q}[x, 1/x, 1/r])$ denote a matrix such that $[b^\infty_0, \ldots, b^\infty_{d_y-1}]$ is an integral basis for $\mathbb{Q}(\mathcal{X})$ over $\mathbb{Q}[1/x]$. 
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Example

When the plane model $\mathbb{Q}(x, y) = 0$ is smooth, we can take $W^0 = I$ since $[y^0, \ldots, y^{d_y-1}]$ is already an integral basis in that case.
Good reduction at $p$

We need to impose some conditions on the prime $p$:

\begin{enumerate}
  \item the curve $X$ has good reduction at $p$,
  \item the divisors defined by $r(x)$ on $X$ and on $\mathbb{P}^1$ have good reduction at $p$, i.e., the points in their support all have different reductions modulo $p$.
  \item $W_0 \in \text{Gl}_d(\mathbb{Z}_p[1/r])$,
  \item $W_\infty \in \text{Gl}_d(\mathbb{Z}_p[1/x, 1/r])$.
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We say that the triple $(\mathcal{Q}, W^0, W^\infty)$ has good reduction at a prime number $p$, if the following conditions hold:

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Remark $(\mathcal{Q}, W^0, W^\infty)$ has good reduction at all but a finite number of primes $p$ and for Chabauty one can vary $p$. However, for computing zeta functions $p$ is fixed and it can in general be hard to find a lift that has good reduction in the above sense.
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$(Q, W^0, W^\infty)$ has good reduction at all but a finite number of primes $p$ and for Chabauty one can vary $p$. However, for computing zeta functions $p$ is fixed and it can in general be hard to find a lift that has good reduction in the above sense.
Overconvergent rings

From now on we assume that \((\mathbb{Q}, \mathcal{W}^0, \mathcal{W}^\infty)\) has good reduction at \(p\) and denote \(X = \mathcal{X} \otimes \mathbb{Z}_p\).

Let:

- \(V\) the Zariski open of \(\mathbb{P}^1_{\mathbb{Z}_p}\) defined by the two conditions \(x \neq \infty\) and \(r(x) \neq 0\),
- \(U = x^{-1}(V)\) the Zariski open of \(X\) lying over \(V\),
- \(\overline{X}, \overline{U}\) and \(\overline{V}\) the reductions modulo \(p\) of \(X, U\) and \(V\), respectively.
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- $\overline{X}, \overline{U}$ and $\overline{V}$ the reductions modulo $p$ of $X, U$ and $V$, respectively.

We write

$$S^\dagger = \mathbb{Q}_p \langle x, 1/r \rangle^\dagger, \quad R^\dagger = \mathbb{Q}_p \langle x, 1/r, y \rangle^\dagger / (Q).$$

where $\langle \rangle^\dagger$ denotes weak completion, i.e.

$$\mathbb{Q}_p \langle x_1, \ldots, x_m \rangle^\dagger = \left\{ \sum_I c_I x_1^{i_1} \ldots x_m^{i_m} : \text{radius of convergence} > 1 \right\}.$$
Lifting Frobenius

The $p$-the power Frobenius map $F_p$ in characteristic $p$ can be lifted to the rings $S^\dagger = \mathbb{Q}_p \langle x, 1/r \rangle^\dagger$ and $R^\dagger = \mathbb{Q}_p \langle x, 1/r, y \rangle^\dagger / (Q)$ in the following way:

- Set $F_p(x) = x^p$.
- Compute $F_p(1/r) \in S^\dagger$ Hensel lifting $F_p(1/r) = 1/r(x^p)$, starting from $1/r^p$.
- Compute $F_p(y) \in R^\dagger$ Hensel lifting $Q(x^p, F_p(y)) = 0$, starting from $y^p$.

Remark
In practice it is important that $F_p(x) = x^p$. However, for a Frobenius lift of this form to exist, we need to remove the zeros of $r(x)$ from the curve.
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Remark

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The p-adic cohomology of \( \overline{U} \) is the cohomology of the overconvergent de Rham complex \( \Omega^\bullet_{R^\dagger} \). More precisely, we have \( \Omega^1_{R^\dagger} = R^\dagger dx \oplus R^\dagger dy \) and

\[
H^0_{\text{rig}}(U) = \ker(d : R^\dagger \to \Omega^1_{R^\dagger}),
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H^1_{\text{rig}}(U) = \text{coker}(d : R^\dagger \to \Omega^1_{R^\dagger}).
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By the assumption on good reduction, there is a comparison theorem with algebraic De Rham cohomology:

$$H^i_{\text{rig}}(\overline{U}) \cong H^i_{dR}(U \otimes \mathbb{Q}_p) \text{ for } i = 0, 1.$$
**p-adic cohomology**

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**Theorem**

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**Remark**

We can define $H^1_{\text{rig}}(\overline{X}) \subset H^1_{\text{rig}}(\overline{U})$ as the kernel of a residue map.
Reducing in cohomology

Proposition

For all \( \ell \in \mathbb{N} \) and every vector \( w \in \mathbb{Q}_p[x] \oplus d_y \), there exist vectors \( u, v \in \mathbb{Q}_p[x] \oplus d_y \) with \( \deg(v) < \deg(r) \), such that

\[
\sum_{i=0}^{d_y-1} w_i b_i^0 \frac{dx}{r^{\ell}} = d \left( \sum_{i=0}^{d_y-1} v_i b_i^0 \frac{dx}{r^{\ell}} \right) + \sum_{i=0}^{d_y-1} u_i b_i^0 \frac{dx}{r^{\ell-1}}.
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Reduction in cohomology

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\sum_{i=0}^{d_y-1} w_i b_i^0 \frac{dx}{r^\ell} = d \left( \frac{\sum_{i=0}^{d_y-1} v_i b_i^0}{r^\ell} \right) + \frac{\sum_{i=0}^{d_y-1} u_i b_i^0}{r^{\ell-1}} \frac{dx}{r}.
\]

**Idea of proof.**

To lowest order in \( r \), the vector \( v \) has to satisfy the \( d_y \times d_y \) linear system

\[
\left( \frac{rG^0}{r'} - \ell I \right) v \equiv \frac{w}{r'} \pmod{r}
\]

over \( \mathbb{Q}_p[x]/(r) \) for some matrix \( G^0 \in M_{d_y \times d_y}(\mathbb{Q}_p[x]) \) such that the eigenvalues of \( \frac{rG^0}{r'} \) are contained in \( \mathbb{Q} \cap [0, 1) \cap \mathbb{Z}_p \) at every zero of \( r(x) \). Therefore, as long as \( \ell \geq 1 \) we can solve the system and reduce the pole order at the zeros of \( r(x) \). \( \square \)
Computing the cohomology

In these reductions we have used that \([b_0^0, \ldots, b_{d_y-1}^0]\) is an integral basis for \(\mathbb{Q}(X)\) over \(\mathbb{Q}[x]\), otherwise \(G^0\) would not consist of polynomials.

By applying repeatedly, we can represent the cohomology class of any 1-form on \(U\) by one that is logarithmic at all \(P \in X \setminus U\) for which \(x(P) \neq \infty\).
Computing the cohomology

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By applying repeatedly, we can can represent the cohomology class of any 1-form on $U$ by one that is logarithmic at all $P \in X \setminus U$ for which $x(P) \neq \infty$.

We can do something similar at the points $P$ with $x(P) = \infty$ by working with the integral basis $[b_0^\infty, \ldots, b_{d_y-1}^\infty]$ of $\mathbb{Q}(X)$ over $\mathbb{Q}[1/x]$.

Finding a basis for $H^1_{\text{rig}}(\overline{U})$ is now reduced to finite dimensional linear algebra.

We find 1-forms $\omega_1, \ldots, \omega_\kappa$ in $\Omega^1(U)$ that are a basis for $H^1_{\text{rig}}(\overline{U})$ such that the first $2g$ are a basis for $H^1_{\text{rig}}(X)$. 
By applying $F_p$ and using the cohomological reductions, we find a matrix $\Phi \in M_{\kappa \times \kappa}(\mathbb{Q}_p)$ and functions $f_1, \ldots, f_\kappa \in R^\dagger$ such that:

$$F_p^*(\omega_i) = df_i + \sum_j \Phi_{ij}\omega_j$$

for $i = 1, \ldots, \kappa$. 
Computing matrix of Frobenius

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$$F_p^*(\omega_i) = df_i + \sum_j \Phi_{ij} \omega_j$$

for $i = 1, \ldots, \kappa$.

$\Phi$ is the matrix of Frobenius on $H^1_{\text{rig}}(\overline{U})$ w.r.t. the basis $[\omega_1, \ldots, \omega_{\kappa}]$.

Before we did not care about $f_1, \ldots, f_{\kappa}$ and computed the zeta function of $\overline{X}$ as the reverse characteristic polynomial of the matrix $\Phi$.

Now we are going to compute Coleman integrals on $X$ using $\Phi$ and $f_1, \ldots, f_{\kappa}$.
Residue disks

There is a specialisation map from the analytic space $X^{an}$ over $\mathbb{Q}_p$ to $\overline{X}$ that should be seen as reduction mod $p$.

The inverse image of a point on $\overline{X}$ under this map is called a residue disk and is isomorphic to the open unit disk $|z| < 1$.
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We call a residue disk bad if it contains a point of $X \setminus U$ and good if not.

Similarly, we say that a bad residue disk is infinite if it contains a point $P$ with $x(P) = \infty$ and finite if not.
Tiny integrals

Suppose that $P, Q \in X(Q_p)$ are points in the same residue disk $D$ and $\omega \in \Omega^1(U)$.

For simplicity, assume that $\omega$ does not have a pole on $D$, for example because $D$ is a good disk.
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Then $\int_P^Q \omega$ can be computed simply by expanding $\omega$ in terms of a local coordinate $t$ on the disk:

$$\omega = \sum_{i \geq 0} c_i t^i dt$$

and integrating as usual

$$\int_{t(P)}^{t(Q)} \sum_{i \geq 0} c_i t^i dt = \sum_{i \geq 0} \frac{c_i}{i+1} (t(Q)^{i+1} - t(P)^{i+1}).$$

This is the easy case, no $p$-adic cohomology is needed.
Tiny integrals: precision

Proposition

Suppose that $P$, $Q$ and $\omega$ are accurate to $p$-adic precision $N$. If we assume that $\omega \in \mathbb{Z}_p[[t]]$ and truncate it modulo $t^m$, then the tiny integral as computed above is accurate to $p$-adic precision

$$\min\{N, m + 1 - \lfloor \log_p(m + 1) \rfloor \}.$$
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Proof.

Let us denote the $i$-th term by $T_i = \frac{c_i}{i+1} (t(Q)^{i+1} - t(P)^{i+1})$. The effect of the truncation is to omit the $T_i$ for $i \geq m$. However, $\text{ord}_p(t(Q)), \text{ord}_p(t(P)) \geq 1$, so for $i \geq m$ we have

$$\text{ord}_p(T_i) \geq i + 1 - \lfloor \log_p(i + 1) \rfloor \geq m + 1 - \lfloor \log_p(m + 1) \rfloor.$$ 

Since $t(P), t(Q)$ are accurate to $p$-adic precision $N$, for $i < m$ we have that $T_i$ is accurate to precision

$$N + i - \lfloor \log_p(i + 1) \rfloor \geq N.$$
Good endpoints

Now suppose that $P, Q \in X(Q_p)$ are points in different good residue disks.

We may assume that $P, Q$ are Teichmüller points (fixed under $F_p$), because the integral from a point to the corresponding Teichmüller point is tiny!
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Recall that for $i = 1, \ldots, \kappa$

$$F_p^*(\omega_i) = df_i + \sum_j \Phi_{ij}\omega_j.$$
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Integrating, we find

$$\int_P^Q \omega_i = \int_{F_p(P)}^{F_p(Q)} \omega_i = \int_P^Q F_p^*(\omega_i) = f_i(Q) - f_i(P) + \sum_j \Phi_{ij} \int_P^Q \omega_j.$$

So we can find the $\int_P^Q \omega_i$ by solving the linear system

$$(\Phi - I) \int_P^Q \omega_i = f_i(P) - f_i(Q).$$
Good endpoints: precision

Proposition

Suppose that \( P, Q \in X(Q_p) \) are points lying in good disks, accurate to \( N \) digits of precision, and suppose that the matrix \( \Phi \) and the functions \( f_i \) are accurate to \( N \) digits of precision as well. Then the computed values of \( \int_P^Q \omega_i \) will be accurate to \( N - \text{ord}_p(\det(\Phi - I)) \) digits of precision.
Good endpoints: precision

Proposition

Suppose that $P, Q \in X(Q_p)$ are points lying in good disks, accurate to $N$ digits of precision, and suppose that the matrix $\Phi$ and the functions $f_i$ are accurate to $N$ digits of precision as well. Then the computed values of $\int_P^Q \omega_i$ will be accurate to $N - \text{ord}_p(\det(\Phi - I))$ digits of precision.

Proof.

The evaluation of the $f_i$ at $P, Q$ does not suffer from precision loss, since $P, Q$ lie in good disks! The matrix inversion loses at most $\text{ord}_p(\det(\Phi - I))$ digits of precision.
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Remark

Note that we can integrate any $\omega \in \Omega^1(U)$ using

$$\int_P^Q \omega = \int_P^Q (df + \sum_i c_i \omega_i) = f(Q) - f(P) + \sum_i c_i \int_P^Q \omega_i.$$
Suppose that $P \in \mathcal{X}(Q_p)$ lies in a good disk, but $Q \in \mathcal{X}(Q_p)$ lies in a finite bad disk $D$ (the case of an infinite bad disk is easier).
Bad endpoints

Suppose that $P \in X(Q_p)$ lies in a good disk, but $Q \in X(Q_p)$ lies in a finite bad disk $D$ (the case of an infinite bad disk is easier).

Now the problem is that the $f_i$ will in general have a pole in $D$, so that $f_i(Q)$ does not necessarily converge!

However, the $f_i$ will converge close enough to $\partial D$. Therefore, we compute $\int_{Q'} P \omega_i$ for some $Q'$ close enough to $\partial D$. Note that $\int_{Q'} Q \omega_i$ is a tiny integral!

How close is close enough?
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How close is close enough?
Bad endpoints: convergence

Proposition

On a finite bad disk, the functions $f_i$ converge outside of the closed disk defined by $\text{ord}_p(r(x)) \geq 1/p$. 
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Proof.

The part of $f_i$ coming from finite reductions is of the form

$$\sum_{j=0}^{d_y-1} \sum_{k=1}^{\infty} \frac{c_{ijk}(x)}{r(x)^k} b_j,$$

with the $c_{ijk}$ elements of $\mathbb{Q}_p[x]$ of degree smaller than $\text{deg}(r)$, that satisfy

$$\text{ord}_p(c_{ijk}) \geq \left\lfloor \frac{k}{p} \right\rfloor + 1 - \left\lfloor \log_p(k e_0) \right\rfloor$$

where $e_0 = \max \{ e_P : P \text{ bad finite point} \}$ and $e_P$ denotes the ramification index of $x$ at $P$. It is therefore clear that the series converges if $\text{ord}_P(r(x)) < 1/p$. ∎
Proposition

Suppose that $P \in X(\mathbb{Q}_p)$ is a point lying in a good disk, and $Q \in X(\mathbb{Q}_p(p^{1/m}))$ for some a point lying in a finite bad disk, both accurate to $N$ digits of precision. Assume that $\Phi$ and the functions $f_i$ are accurate to $N$ digits of precision as well. Denote $\epsilon = \text{ord}_p(r(Q))$ and suppose that $\epsilon < 1/p$. Define a function $\pi$ on positive integers by

$$
\pi(k) = \max\{N, \left\lfloor k/p \right\rfloor + 1 - \left\lfloor \log_p(ke_0) \right\rfloor \},
$$

where $e_0 = \max\{e_P : P \text{ finite bad point}\}$ and $e_P$ denotes the ramification index of $x$ at $P$. (Note that $\pi(k) - k\epsilon \to \infty$ as $k \to \infty$). Then the computed values of $\int_P^Q \omega_i$ will be accurate to

$$
\min_{k \in \mathbb{Z}_{>0}} \{\pi(k) - k\epsilon\} - \text{ord}_p(\det(\Phi - I)).
$$

digits of precision.
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Implementation

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We plan on making the code available, once there is a stable version.
Bruin, Poonen and Stoll (2013) give the example

\[ Q = y^3 + (-x^2 - 1)y^2 - x^3 y + x^3 + 2x^2 + x \]

which has small coefficients and rank 1 (under GRH).

There are 5 finite points

\[ P_1 = (0, 0), \; P_2 = (0, 1), \; P_3 = (-3, 4), \; P_4 = (-1, 0), \; P_5 = (-1, 1) \]

and 3 more at infinity.
Some integrals

We computed the integrals at $p = 3$ of $\omega_1, \ldots, \omega_6$ between $P_1$ and $P_2, P_3, P_4$:

\[
\left( \int_{P_1}^{P_2} \omega_i \right)_{i=1,\ldots,6} = (-808a^{60} + O(a^{280}), 347a^{90} + O(a^{280}), -1646a^{60} + O(a^{280}), \\
3667a^{30} + O(a^{280}), 3172a^{30} + O(a^{280}), -5164a^{30} + O(a^{280}))
\]

\[
\left( \int_{P_1}^{P_3} \omega_i \right)_{i=1,\ldots,6} = (1690a^{60} + O(a^{280}), 319a^{90} + O(a^{280}), -1072a^{60} + O(a^{280}), \\
7474a^{30} + O(a^{280}), 3022a^{30} + O(a^{280}), 3509a^{30} + O(a^{280}))
\]

\[
\left( \int_{P_1}^{P_4} \omega_i \right)_{i=1,\ldots,6} = (-55a^{90} + O(a^{280}), -349a^{120} + O(a^{280}), 229a^{90} + O(a^{280}), \\
-4918a^{30} + O(a^{280}), 565a^{90} + O(a^{280}), 8507a^{30} + O(a^{280}))
\]

where $a^{30} = 3$ ($a$ is needed for things to converge in the bad disks).
Chabauty

We now compute independent linear combinations $\xi_1, \xi_2$ of $\omega_1, \omega_2, \omega_3$ such that

$$\int_{P_1}^{P_2} \xi_i = \int_{P_1}^{P_3} \xi_i = \int_{P_1}^{P_4} \xi_i = 0 \quad \text{for } i = 1, 2$$

and find

$$\begin{align*}
\xi_1 &= (1 + O(3^8))\omega_1 + O(3^{10})\omega_2 + (430 + O(3^8))\omega_3 \\
\xi_2 &= O(3^8)\omega_1 + (1 + O(3^{10}))\omega_2 + (-320 \cdot 3 + O(3^8))\omega_3.
\end{align*}$$

No we check that

$$\int_{P_1}^{P_5} \xi_1 = \int_{P_1}^{P_5} \xi_2 = 0.$$

So we must be doing something right!