Explicit Chabauty and the cursed curve

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§1 Introduction

$E/\mathbb{Q}$ elliptic curve $\rightsquigarrow$ $\rho_{E,\ell} : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to Aut(E[\ell]) \simeq GL_2(\mathbb{F}_\ell)$
ℓ prime

Thm (Serre '72)
If E does not have CM then $\rho_{E,\ell}$ is surjective for $\ell \gg 0$

Question (Serre) uniformity?
Is there a constant ℓ_0 s.t $\rho_{E,\ell}$ is surjective for all $\ell > \ell_0$ and all $E/\mathbb{Q}$ without CM?

Idea. If $Im(\rho_{E,\ell}) \neq GL_2(\mathbb{F}_\ell)$ then contained in a maximal proper subgroup of $GL_2(\mathbb{F}_\ell)$. These are of the form:

(i) Borel
(ii) exceptional
(iii) normaliser split Cartan $C_s(\ell)^+$
(iv) normaliser non-split Cartan.

Answer to Serre's question positive for (i) (Mazur), (ii) (Serre), (iii) (Bilu - Parent '11) and wide open for (iv).

can be more precise:
Thm (Bilu - Parent - Rebolledo '13)
$E/\mathbb{Q}$ elliptic curve without CM, $\ell > 7$ and $\ell \neq 13 \Rightarrow Im\ \rho_{E,\ell} \not\subset C_s(\ell)^+$

Idea

$E/\mathbb{Q}$ with $\rho_{E,\ell} \subset C_s(\ell)^+$ correspond to $P_E \in X_s(\ell)(\mathbb{Q})$
where $X_s(\ell) = X(\ell)/C_s(\ell)^+$.

$J_s(\ell) = \mathrm{Jac}(X_s(\ell))$, $J_0(\ell) = \mathrm{Jac}(X_0(\ell))$, $J_{ns}(\ell) = \mathrm{Jac}(X_{ns}(\ell))$

one can show that $J_s(\ell) \sim J_0(\ell) \times J_{ns}(\ell)$

$$\begin{cases} \ell > 13 \Rightarrow J_0(\ell) \neq 0 \rightsquigarrow \text{Mazur's method} \\ \ell = 13 \Rightarrow J_0(13) = 0 \rightsquigarrow \text{does not work : } \underline{\text{cursed curve}} \end{cases}$$

Facts about $X_s(13)$:

- genus $g = 3$, non hyp
- 7 known rational points (6 CM and 1 cusp)
- ($\rho = \mathrm{rk}\,(NS(J_s(13))) = 3$ CRM by $\mathbb{Q}(\zeta_7^+)$)
- $r = \mathrm{rk}\, J_s(13)(\mathbb{Q}) = 3$
- potentially good reduction everywhere.
- (equation: $y^4 + 5x^4 - 6x^2y^2 + 6x^3 + 26x^2y + 10xy^2 - 10y^3 - 32x^2 - 40xy + 24y^2 + 32x - 16y = 0$)

Thm (BDMTV)

The only rational points on $X_s(13)$ are the 7 known ones.

Cor

There is no $E/\mathbb{Q}$ without CM s.t. $\mathrm{Im}\,\rho_{E,13} \subset C_s(13)^+$.

§2 The Chabauty method

Let:

- $X/\mathbb{Q}$ smooth proj geom. int. curve
- genus $g > 1$
- $r = \mathrm{rk}\,(\mathrm{Jac}(X)(\mathbb{Q}))$
- $b \in X(\mathbb{Q}) \neq \emptyset$, $X \hookrightarrow \mathrm{Jac}(X)$ by $p \longmapsto (p) - b$
- p prime of good reduction

Thm (Chabauty-Coleman)

Suppose that $r < g$

Then $\exists\, \omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1)$ s.t. $\int_b^P \omega = 0 \quad \forall P \in X(\mathbb{Q})$

sketch of proof:

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook] \arrow[d] & X(\mathbb{Q}_p) \arrow[d] \arrow[dr, "AJ_b"] & \\
J(\mathbb{Q}) \arrow[r, hook] & J(\mathbb{Q}_p) \arrow[r, "D \mapsto \int_D"] & H^0(X_{\mathbb{Q}_p}, \Omega^1)^*
\end{tikzcd}$$

$X(\mathbb{Q})$ lands in a subspace of $H^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ of dim $\leq r$

can (sometimes) be made effective, provably determine all pts

see: github.com/jtuitman/Coleman for many examples.

But For $X_S(13)$ $r = g = 3$, so does not work

$D \mapsto \int_D$ gives isomorphism $J(\mathbb{Q}) \otimes \mathbb{Q}_p \xrightarrow{\simeq} H^0_{\cancel{dR}}(X_{\mathbb{Q}_p}, \Omega^1)^*$

§3 Quadratic Chabauty

so cannot find $\mathbb{Q}$-points among $\mathbb{Q}_p$-points with linear relations in Abel Jacobi map AJ_b.

Idea (M. Kim, '06 -)
Replace linear relations by higher degree ones (bilinear ones for us).

From now on suppose X has pot good reduction everywhere (like $X_s(13)$).

Def A quadratic Chabauty function is $\theta : X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ such that:

① on each residue disk the map:

$$(AJ_b, \theta) : X(\mathbb{Q}_p) \longrightarrow H^0(X_{\mathbb{Q}_p}, \Omega^1)^* \times \mathbb{Q}_p$$

has Zariski dense image and is given by power series.

② there exist:

- endomorphism E of $H^0(X_{\mathbb{Q}_p}, \Omega^1)^*$
- constant $c \in H^0(X_{\mathbb{Q}_p}, \Omega^1)^*$
- bilinear form $B : H^0(X_{\mathbb{Q}_p}, \Omega^1)^* \otimes H^0(X_{\mathbb{Q}_p}, \Omega^1)^* \longrightarrow \mathbb{Q}_p$

such that, for all $x \in X(\mathbb{Q})$:

$$\theta(x) = B\big(AJ_b(x), E(AJ_b(x)) + c\big)$$

How to find such a θ? From a nice correspondence.

Def

Let Z be a correspondence on X, i.e divisor on $X \times X$

denote
- $\tau : (x_1, x_2) \longmapsto (x_2, x_1)$ involution
- $\pi_1, \pi_2 : X \times X \longrightarrow X$ projections

Z is symmetric if $\exists\, Z_1, Z_2 \in \mathrm{Pic}(X)$ s.t

$$\tau_* Z = Z + \pi_1^*(Z_1) + \pi_2^*(Z_2)$$

induces endomorphism ξ_Z of $H^1_{dR}(X)$ and class in $H^1_{dR}(X_{\mathbb{Q}_p}) \otimes H^1_{dR}(X_{\mathbb{Q}_p})$

Z is nice if:
- nontrivial
- symmetric
- $\mathrm{Tr}(\xi_Z) = 0$

Rem for $X_S(13)$ will use Hecke correspondences.

construction of θ_Z associated to Z: (fix Z)

Let $Y = X - x^{-1}(\infty)$

$\vec{\omega} = \omega_1, \dots, \omega_6$ basis of $= H^1_{dR}(X_{\mathbb{Q}_p})$

class of Z : $\sum Z_{ij}\, \omega_i \otimes \omega_j \qquad Z_{ij} \in M_{2g \times 2g}(\mathbb{Q}_p)$

Put connection $\nabla = d + \Lambda$ on $\mathcal{A}_Z(b) = \mathcal{O}_Y \oplus \mathcal{O}_Y^{\oplus 6} \oplus \mathcal{O}_Y$ where

$$\Lambda = -\begin{pmatrix} 0 & 0 & 0 \\ \vec{\omega} & 0 & 0 \\ \eta & \vec{\omega}^t Z & 0 \end{pmatrix}$$

and η logarithmic s.t ∇ extends holomorphically to X.

By crystalline comparison thm, $(\mathcal{A}_Z(b), \nabla)$ admits a Frobenius structure:

$$F : F_p^* \mathcal{A}_Z(b) \longrightarrow \mathcal{A}_Z(b)$$

turning $(\mathcal{A}_Z(b), \nabla)$ into unipotent overconvergent F-isocrystal.

Let b_0 Teichmüller lift of b i.e $\begin{cases} F_p(b_0) = b_0 \\ b_0 \equiv b \mod p \end{cases}$

inverse of matrix of Frobenius structure F given by:

$$\mathcal{G} = \begin{pmatrix} 1 & 0 & 0 \\ \vec{f} & \Phi & 0 \\ h & \vec{g}^t & p \end{pmatrix} \qquad \text{s.t} \quad d\mathcal{G} = \mathcal{G}\Lambda - F_p^*(\Lambda)\mathcal{G}$$

Differential equation equiv to:

$$\left(\begin{array}{ll} F_p^* \omega = d\vec{f} + \Phi \vec{\omega} & \vec{f}(b^0) = 0 \\ d\vec{g}^t = d\vec{f}^t Z \Phi & \\ dh = \vec{\omega}^t \Phi^t Z \vec{f} + d\vec{f}^t Z \vec{f} - \vec{g}^t \vec{\omega} + F_p^* \eta - p\eta & \vec{h}(b^0) = 0 \end{array} \right)$$

can be solved using my algorithms ☺

For any $x \in X(\mathbb{Q}_p)$ can pullback $A_Z(b)$:

$$A_Z(b, x) = x^*(A_Z(b))$$

mixed extension of filtered ϕ-modules in sense of p-adic Hodge theory.

action of ϕ given by:

$$T_{x_0, x} \circ \mathcal{G}^{-1}(x_0) \circ T_{b, b_0}$$

where $T_{x,y}$ is parallel transport from x to y.

For such objects, Nekovar defines a p-adic height function $h_p(\ )$

We set:

$$\theta_Z(x) = h_p(A_Z(b, x)).$$

For any nice correspondence Z, θ_Z is quadratic Chabauty function. ($E = \xi_Z$, c explicit as well).